Linear-complexity CPHD filters

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Abstract* – *The probability hypothesis density (PHD) filter and cardinalized probability hypothesis density (CPHD) filter are principled approximations of the general multitarget Bayes recursive filter. If n is the current number of tracks and m the current number of measurements, then the former has computational complexity $O(mn)$ and the latter $O(m^3n)$. Although the CPHD filter has better target detection and tracking performance than the PHD filter, its cubic complexity in m will limit its practicality in many applications. The purpose of this paper is to derive a CPHD filter that, thanks to a simplified clutter model, has computational complexity $O(mn)$. I also show how to extend this simplified CPHD filter to the multisensor case.

Keywords: CPHD filter, PHD filter, multitarget tracking, multitarget filtering, random sets, multisource integration.

1 Introduction

The probability hypothesis density (PHD) filter [6,8] and cardinalized PHD (CPHD) filter [7,8] are principled approximations of the single-sensor, multitarget recursive Bayes filter. The PHD filter propagates a probability hypothesis density (PHD) $D_{k|k}(\mathbf{x}|Z^{(k)})$ with computational complexity $O(mn)$, where m is the current number of measurements and n the current number of tracks. Although computationally attractive, it provides large-variance instantaneous estimates of target number.

The cardinalized PHD (CPHD) filter addresses this shortcoming. In addition to the PHD, it propagates the distribution $p_{k|k}(n|Z^{(k)})$ on target number n (a.k.a. the cardinality distribution). Although it has better tracking performance than the PHD filter, it achieves this at the price of greater computational complexity: $O(m^3n)$. The cubic complexity in m will limit the practicality of the CPHD filter in many applications.

In this paper I derive a *linear-complexity CPHD* (LC-CPHD) *filter*—i.e., a filter that propagates $D_{k|k}(\mathbf{x}|Z^{(k)})$ and $p_{k|k}(n|Z^{(k)})$ with complexity $O(mn)$. I also derive multisensor versions of this filter.

The single-sensor LC-CPHD filter is a special case of CPHD filters that are "clutter-agnostic"—i.e., capable of operation in unknown, dynamic clutter backgrounds.

In [3] I assumed that clutter measurements are generated by an unknown number of clutter generators, with the measurement process for each generator being a Poisson process with unknown parameters. This filter appears to be computationally intractable.

In a subsequent paper [9] I achieved computational tractability by using a simpler clutter model. The measurement process for each clutter generator $\mathbf{c}$ is a Bernoulli process with unknown parameters. That is: each clutter generator generates at most a single measurement with probability $0 \leq c \leq 1$, and the spatial distribution of this measurement is $c_{k+1}(\mathbf{z}|\mathbf{c})$.

The LC-CPHD filter results when we assume that (1) $c = 1$ and $c_{k+1}(\mathbf{z}|\mathbf{c}) = c_{k+1}(\mathbf{z})$, (2) no clutter generators survive prediction, (3) clutter generators spontaneously appear during prediction, and (4) the birth process for these generators is a known i.i.d. cluster process.

1.1 Organization of the paper

The paper is organized as follows. I summarize the main results in Section 2 (single-sensor case) and Section 3 (multisensor case). In support of the derivation, in Sections 4 and 5 I summarize, respectively, "clutter-agnostic" CPHD filters [9] and the multisensor CPHD filter [1]. In Section 6 I show that the "multitarget intensity filter," claimed to be a clutter-agnostic generalization of the PHD filter, is profoundly erroneous. I also point out that the most general "clutter-agnostic" PHD filter described in [9] is a non-erroneous, theoretically rigorous "intensity filter." Mathematical derivations are in Section 7 and conclusions in Section 8.

2 Main results: Single-sensor

In this section I summarize the predictor equations (Section 2.1), corrector equations (Section 2.2), and multitarget state estimation procedure (Section 2.3) for the single-sensor LC-CPHD filter. The abbreviation p.g.f. means "probability generating function."

2.1 Predictor equations: single-sensor

In what follows, let:

- $b_{k+1|k}(\mathbf{x})$ = PHD of the target-appearance process,
- $p_{S,k+1|k}(\mathbf{x}')$ = probability that a target with state $\mathbf{x}'$ at time-step k will survive into time-step k+1,
- $f_{k+1|k}(\mathbf{x}|\mathbf{x}')$ = single-target Markov transition density,
- $B_{k+1|k}(x)$ = p.g.f. for the target-birth process,
- $C_{k+1}(z)$ = p.g.f. of the clutter process at time-step k+1, with clutter rate $C^{(1)}_{k+1}(1) = \lambda_{k+1}$.

Then the predictor equation for the PHD is

$$D_{k+1|k}(\mathbf{x}) = b_{k+1|k}(\mathbf{x}) + \int p_{S,k}(\mathbf{x}') \cdot f_{k+1|k}(\mathbf{x} \mid \mathbf{x}') \cdot D_{k|k}(\mathbf{x}) d\mathbf{x} \quad (1)$$

The predictor equation for the p.g.f. is

$$G_{k+1|k}(x) = B_{k+1|k}(x) \cdot C_{k+1}(x) \cdot G_{k|k}(1 - \psi_k + \psi_k \cdot x) \quad (2)$$

where

$$\psi_k = \frac{D_{k|k}[p_{S,s+1|k}]}{G^{(1)}_{k|k}(1)} \quad (3)$$

and where $D_{k|k}[h] = \int h(\mathbf{x}) \cdot D_{k|k}(\mathbf{x}) d\mathbf{x}$ for any $h(\mathbf{x})$. Eqs. (1-3) are established in Section 7.1.

2.2 Corrector equations: single-sensor

Let a single sensor collect measurements. Let:

- $p_D(\mathbf{x})$ = probability of detection at time-step k+1,
- $L_\mathbf{z}(\mathbf{x}) = f_{k+1}(\mathbf{z}|\mathbf{x})$ = likelihood function at time-step k+1,
- $c_{k+1}(\mathbf{z})$ = clutter spatial distribution at time-step k+1,
- λ_{k+1} = clutter rate at time-step k+1.

Assume $D_{k+1|k}[p_D L_\mathbf{z}] > 0$ for all $\mathbf{z}$. The sensor collects a measurement-set Z_{k+1} with $|Z_{k+1}| = m$. Then

$$\frac{D_{k+1|k+1}(\mathbf{x})}{D_{k+1|k}(\mathbf{x})} = \frac{1 - p_D(\mathbf{x})}{N_{k+1|k} + \lambda_{k+1}} \cdot \frac{G^{(m+1)}_{k+1|k}(\phi_k)}{G^{(m)}_{k+1|k}(\phi_k)} + \sum_{\mathbf{z} \in Z_{k+1}} \frac{p_D(\mathbf{x}) \cdot L_\mathbf{z}(\mathbf{x})}{\lambda_{k+1} c_{k+1}(\mathbf{z}) + D_{k+1|k}[p_D L_\mathbf{z}]} \quad (4)$$

where

$$\phi_k = \frac{D_{k+1|k}[1 - p_D]}{N_{k+1|k} + \lambda_{k+1}} \quad (5)$$

Also, the corrector equation for the p.g.f. is

$$G_{k+1|k+1}(x) = \frac{x^m \cdot G^{(m)}_{k+1|k}(x \cdot \phi_k)}{G^{(m)}_{k+1|k}(\phi_k)} \quad (6)$$

In Section 3.2 we will also need the corrector equation for the cardinality distribution. This is

$$p_{k+1|k+1}(n) = \frac{\ell_{Z_{k+1}}(n) \cdot p_{k+1|k}(n)}{\sum_{j \geq 0} \ell_{Z_{k+1}}(j) \cdot p_{k+1|k}(j)} \quad (7)$$

where

$$\ell_{Z_{k+1}}(n) = C_{n,m} \cdot \phi_k^{n-m} \quad (8)$$

and where $C_{n,m}$ is the combinatorial coefficient, with $C_{n,m} = 0$ if $n < m$. Eqs. (4-8) are proved in Section 7.2.

2.3 Multitarget state estimation

The usual procedure for multitarget state estimation with the CPHD filter is: (1) determine the MAP estimate $\nu = \text{argsup}_n \, p_{k+1|k+1}(n)$ of target number; (2) determine the ν largest peaks of the PHD $D_{k+1|k+1}(\mathbf{x})$; and (3) determine the states $\mathbf{x}_1, \ldots, \mathbf{x}_\nu$ corresponding to these peaks. Then $X_{k+1|k+1} = \{\mathbf{x}_1, \ldots, \mathbf{x}_\nu\}$ is the multitarget state estimate.

For the LC-CPHD filter, this procedure is invalid. The reason is that n in $p_{k+1|k+1}(n)$ is the sum of the number of targets and the number of clutter generators. Step (1) is replaced by: compute the expected number of targets, $N_{k+1|k+1} = D_{k+1|k+1}(\mathbf{x}) d\mathbf{x}$, and round off $N_{k+1|k+1}$ to the nearest integer ν. Then proceed as in Steps (2), (3).

3 Main results: Multisensor

In this section I summarize the predictor equations (Section 3.1) and corrector equations (Section 3.2) for the multisensor LC-CPHD filter.

3.1 Predictor equations: multisensor

Suppose that there are s independent sensors and that, for j = 1,…,s, the j^{th} sensor has collected the sequence

$$\overset{j}{Z}{}^{(k)} : \overset{j}{Z}_1, \ldots, \overset{j}{Z}_k \quad (9)$$

of measurement-sets. The predicted PHD and p.g.f.

$$\overset{1\ldots s}{D}_{k+1|k}(\mathbf{x}), \quad \overset{1\ldots s}{G}_{k+1|k}(x) \quad (10)$$

are determined as follows. Let:

- $b_{k+1|k}(\mathbf{x})$ = PHD of the target-appearance process,
- $p_{S,k+1|k}(\mathbf{x}')$ = probability that a target with state $\mathbf{x}'$ at time-step k will survive into time-step k+1,
- $f_{k+1|k}(\mathbf{x}|\mathbf{x}')$ = single-target Markov transition density,
- $B_{k+1|k}(x)$ = p.g.f. for the target-birth process,
- $\overset{j}{C}_{k+1}(z)$ = p.g.f. of the clutter process for the j^{th} sensor at time-step k+1, with clutter rate $\overset{j}{C}{}^{(1)}_{k+1}(1) = \overset{j}{\lambda}_{k+1}$.

Then the predictor equation for the PHD is

$$\overset{1\ldots s}{D}_{k+1|k}(\mathbf{x}) = b_{k+1|k}(\mathbf{x}) + \int p_{S,k}(\mathbf{x}') \cdot f_{k+1|k}(\mathbf{x} \mid \mathbf{x}') \cdot \overset{1\ldots s}{D}_{k|k}(\mathbf{x}) d\mathbf{x} \quad (11)$$

The predictor equation for the p.g.f. is

$$\overset{1\ldots s}{G}_{k+1|k}(x) = B_{k+1|k}(x) \cdot \overset{1}{C}_{k+1}(x) \cdots \overset{s}{C}_{k+1}(x) \cdot \overset{1\ldots s}{G}_{k|k}(1 - \psi_k + \psi_k \cdot x) \quad (12)$$

where

$$\psi_k = \frac{D_{k|k}[p_{S,k+1|k}]}{G^{(1)}_{k|k}(1)} \quad (13)$$

Eqs. (11-13) are established in Section 7.3.

3.2 Corrector equations: multisensor

In what follows, let

$$\overset{1\ldots s}{N}_{k+1|k} = \int \overset{1\ldots s}{D}_{k+1|k}(\mathbf{x}) d\mathbf{x}, \qquad \overset{1\ldots s}{s}_{k+1|k}(\mathbf{x}) = \frac{\overset{1\ldots s}{D}_{k+1|k}(\mathbf{x})}{\overset{1\ldots s}{N}_{k+1|k}} \quad (14)$$

and let $\overset{1\ldots s}{G}_{k+1|k}(x)$ be the p.g.f. of $\overset{1\ldots s}{p}_{k+1|k}(n)$. For j = 1,…,s, let the probability of detection and likelihood function of the j^{th} sensor be

$$\overset{j}{p}_D(\mathbf{x}) = \overset{j}{p}_{D,k+1}(\mathbf{x}), \qquad \overset{j}{L}_{\overset{j}{\mathbf{z}}}(\mathbf{x}) = \overset{j}{f}_{k+1}(\mathbf{z} \mid \mathbf{x}) \quad (15)$$

with $\overset{1\ldots s}{D}_{k+1|k}[\overset{j}{p}_D \overset{j}{L}_\mathbf{z}] > 0$ for all $\overset{j}{\mathbf{z}}$ and all j. Sensor clutter processes are i.i.d. clutter processes where

$$\overset{j}{c}_{k+1}(\overset{j}{\mathbf{z}}), \qquad \overset{j}{C}_{k+1}(z) \quad (16)$$

is the clutter spatial distribution and probability generating function (p.g.f.) of the clutter process.

Finally suppose that, at time-step k+1 and for j = 1,…,s, the sensors simultaneously collect new measurement-sets $\overset{1}{Z}_{k+1}, \ldots, \overset{s}{Z}_{k+1}$ with $|\overset{j}{Z}_{k+1}| = \overset{j}{m}$. Then the measurement-updated multisensor PHD and cardinality distribution are

$$\overset{1\ldots s}{D}_{k+1|k+1}(\mathbf{x}) = \overset{1\ldots s}{L}_{\overset{1}{Z}_{k+1},\ldots,\overset{s}{Z}_{k+1}}(\mathbf{x}) \cdot \overset{1\ldots s}{s}_{k+1|k}(\mathbf{x}) \tag{17}$$

$$\overset{1\ldots s}{p}_{k+1|k+1}(n) = \frac{\tilde{p}(n) \cdot \sigma^n}{\tilde{G}(\sigma)} \tag{18}$$

where

$$\overset{1\ldots s}{L}_{\overset{1}{Z}_{k+1},\ldots,\overset{s}{Z}_{k+1}}(\mathbf{x}) = \frac{\tilde{G}^{(1)}(\sigma)}{\tilde{G}(\sigma)} \cdot \frac{\overset{1}{L}_{\overset{1}{Z}_{k+1}}(\mathbf{x}) \cdots \overset{s}{L}_{\overset{s}{Z}_{k+1}}(\mathbf{x})}{\overset{1}{N}_{k+1|k+1} \cdots \overset{s}{N}_{k+1|k+1}} \tag{19}$$

$$\sigma = \frac{\overset{1\ldots s}{s}_{k+1|k}[\overset{1}{L}_{\overset{1}{Z}_{k+1}} \cdots \overset{s}{L}_{\overset{s}{Z}_{k+1}}]}{\overset{1}{N}_{k+1|k+1} \cdots \overset{s}{N}_{k+1|k+1}} \tag{20}$$

$$\tilde{G}(x) = \sum\nolimits_{n=0}^{\infty} \tilde{p}(n) \cdot x^n \tag{21}$$

$$\tilde{p}(n) = \overset{1}{\ell}_{\overset{1}{Z}_{k+1}}(n) \cdots \overset{s}{\ell}_{\overset{s}{Z}_{k+1}}(n) \cdot \overset{1\ldots s}{p}_{k+1|k}(n) \tag{22}$$

$$\overset{j}{L}_{\overset{j}{Z}_{k+1}}(\mathbf{x}) = \frac{1 - \overset{j}{p}_D(\mathbf{x})}{\overset{1\ldots s}{N}_{k+1|k} + \overset{j}{\lambda}_{k+1}} \cdot \frac{\overset{1\ldots s}{G}{}^{\overset{j}{(m+1)}}_{k+1|k}(\overset{j}{\phi}_k)}{\overset{1\ldots s}{G}{}^{\overset{j}{(m)}}_{k+1|k}(\overset{j}{\phi}_k)} + \sum_{\overset{j}{\mathbf{z}} \in \overset{j}{Z}_{k+1}} \frac{\overset{j}{p}_D(\mathbf{x}) \cdot \overset{j}{L}_{\overset{j}{\mathbf{z}}}(\mathbf{x})}{\overset{j}{\lambda}_{k+1} \overset{j}{c}_{k+1}(\mathbf{z}) + \overset{1\ldots s}{D}_{k+1|k}[\overset{j}{p}_D \overset{j}{L}_{\overset{j}{\mathbf{z}}}]} \tag{23}$$

$$\overset{j}{N}_{k+1|k+1} = \frac{\overset{1\ldots s}{s}_{k+1|k}[1 - \overset{j}{p}_D]}{\overset{1\ldots s}{N}_{k+1|k} + \overset{j}{\lambda}_{k+1}} \cdot \frac{\overset{1\ldots s}{G}{}^{\overset{j}{(m+1)}}_{k+1|k}(\overset{j}{\phi}_k)}{\overset{1\ldots s}{G}{}^{\overset{j}{(m)}}_{k+1|k}(\overset{j}{\phi}_k)} + \sum_{\overset{j}{\mathbf{z}} \in \overset{j}{Z}_{k+1}} \frac{\overset{1\ldots s}{s}_{k+1|k}[\overset{j}{p}_D \overset{j}{L}_{\overset{j}{\mathbf{z}}}]}{\overset{j}{\lambda}_{k+1} \overset{j}{c}_{k+1}(\mathbf{z}) + \overset{1\ldots s}{D}_{k+1|k}[\overset{j}{p}_D \overset{j}{L}_{\overset{j}{\mathbf{z}}}]} \tag{24}$$

and where, for $j = 1,\ldots,s$,

$$\overset{j}{\ell}_{\overset{j}{Z}_{k+1}}(n) = C_{n,\overset{j}{m}} \cdot \overset{j}{\phi}{}_k^n \tag{25}$$

$$\overset{j}{\phi}_k = \frac{\overset{1\ldots s}{D}_{k+1|k}[1 - \overset{j}{p}_D]}{\overset{1\ldots s}{N}_{k+1|k} + \overset{j}{\lambda}_{k+1}} \tag{26}$$

Eqs. (17-26) are established in Section 7.4.

4 Clutter-agnostic CPHD filters (review)

The "classical" PHD and CPHD filters are based on the presumption that the following items are known *a priori*: the global probability of detection $p_D(\mathbf{x})$, and the spatial density $c_{k+1}(\mathbf{z})$ and p.g.f. $C_{k+1}(z)$ of the clutter process. In recent conference papers [2,9,10], I have derived generalizations of the CPHD filter in which

- $p_D(\mathbf{x})$ is unknown ("p_D-agnostic"), or
- $c_{k+1}(\mathbf{z})$ and $C_{k+1}(z)$ are unknown ("clutter-agnostic"), or
- $p_D(\mathbf{x})$, $c_{k+1}(\mathbf{z})$, and $C_{k+1}(z)$ are all unknown ("p_D-and-clutter-agnostic").

There are actually three clutter-agnostic CPHD filters, of successively increasing generality.

The LC-CPHD filter summarized in Section 2.1 is a special case of the least general clutter-agnostic CPHD filter. In this section I describe the clutter-agnostic CPHD filters, with special emphasis on this least general case.

4.1 Clutter-agnostic CPHD filters in general

All three clutter-agnostic CPHD filters assume that the clutter rate

$$\lambda_{k+1} = \int \kappa_{k+1}(\mathbf{z}) d\mathbf{z} \tag{27}$$

is unknown, where $\kappa_{k+1}(\mathbf{z})$ is the PHD (a.k.a. the intensity function) of the clutter process. The first and most general filter assumes that $\kappa_{k+1}(\mathbf{z})$ is completely unknown and must be estimated. The second and less general filter assumes that the spatial distribution

$$c_{k+1}(\mathbf{z}) = \frac{\kappa_{k+1}(\mathbf{z})}{\lambda_{k+1}} \tag{28}$$

is known but that the probability of detection $p_D^\circ(\mathbf{c})$ of any given clutter generator $\mathbf{c}$ is unknown and must be estimated along with λ_{k+1}. The third and least general filter assumes that $c_{k+1}(\mathbf{z})$ is known and that

$$p_D^\circ(\mathbf{c}) = p_D^\circ \tag{29}$$

is known and constant, and that it is λ_{k+1} that must be estimated. This filter has the same computational complexity as a PHD filter: $O(mn)$.

4.2 Simplest clutter-agnostic CPHD filter

This simplest filter [2,9] propagates three items: the target PHD $D_{k|k}(\mathbf{x})$, the expected number $N_{k|k}^\circ$ of clutter generators, and the p.g.f. $G_{k|k}(x)$ of the combined number $\ddot{n} = n + n^\circ$ of targets and clutter generators.

Predictor Equations. The PHD predictor is:

$$D_{k+1|k}(\mathbf{x}) = b_{k+1|k}(\mathbf{x}) + \int p_{S,k+1|k}(\mathbf{x}') \cdot f_{k+1|k}(\mathbf{x} \,|\, \mathbf{x}') \cdot D_{k|k}(\mathbf{x}) d\mathbf{x} \tag{30}$$

where $b_{k+1|k}(\mathbf{x})$ is the PHD of the target-appearance process, $p_{S,k+1|k}(\mathbf{x}')$ is the probability that a target with state $\mathbf{x}'$ at time-step k will survive into time-step k+1, and $f_{k+1|k}(\mathbf{x}|\mathbf{x}')$ is the single-target Markov density.

The predictor equation for the expected number of clutter generators is

$$N_{k+1|k}^\circ = N_{B,k+1|k}^\circ + p_{S,k+1|k}^\circ \cdot N_{k|k}^\circ \tag{31}$$

where $N_{B,k+1|k}^\circ$ is the expected number of appearing clutter generators and $p_{S,k+1|k}^\circ$ is the probability that clutter generators at time-step k will persist into time-step k+1.

The predictor equation for the p.g.f. is

$$G_{k+1|k}(x) = B_{k+1|k}(x) \cdot B_{k+1|k}^\circ(x) \cdot G_{k|k}(1 - \psi_k + \psi_k \cdot x) \tag{32}$$

where $B_{k+1|k}(x)$ is the p.g.f. for the target-birth process, $B_{k+1|k}^\circ(x)$ is the p.g.f. for the generator-birth process, and

$$\psi_k = \frac{D_{k|k}[p_{S,k+1|k}] + p_{S,k+1|k}^\circ N_{k|k}^\circ}{N_{k|k} + N_{k|k}^\circ} \tag{33}$$

Corrector Equations. Let $p_D(\mathbf{x})$ be the probability of detection, $L_{\mathbf{z}}(\mathbf{x}) = f_{k+1}(\mathbf{z}|\mathbf{x})$ the likelihood function, $c_{k+1}(\mathbf{z})$ the clutter spatial distribution, and p_D° the probability of detection for clutter generators. Assume $D_{k+1|k}[p_D L_{\mathbf{z}}] > 0$ for all $\mathbf{z}$. Suppose that a new measurement-set Z_{k+1} is collected with $|Z_{k+1}| = m$. Then the PHD corrector is

$$\frac{D_{k+1|k+1}(\mathbf{x})}{D_{k+1|k}(\mathbf{x})} = \frac{1-p_D(\mathbf{x})}{N_{k+1|k}+N^\circ_{k+1|k}} \cdot \frac{G^{(m+1)}_{k+1|k}(\phi_k)}{G^{(m)}_{k+1|k}(\phi_k)} + \sum_{\mathbf{z}\in Z_{k+1}} \frac{p_D(\mathbf{x})\cdot L_\mathbf{z}(\mathbf{x})}{p^\circ_D N^\circ_{k+1|k} c_{k+1}(\mathbf{z}) + D_{k+1|k}[p_D L_\mathbf{z}]} \tag{34}$$

where

$$\phi_k = \frac{D_{k+1|k}[1-p_D] + N^\circ_{k+1|k}(1-p^\circ_D)}{N_{k+1|k}+N^\circ_{k+1|k}} \tag{35}$$

The corrector for the expected number of clutter generators is

$$\frac{N^\circ_{k+1|k+1}}{N^\circ_{k+1|k}} = \frac{1-p^\circ_D}{N_{k+1|k}+N^\circ_{k+1|k}} \cdot \frac{G^{(m+1)}_{k+1|k}(\phi)}{G^{(m)}_{k+1|k}(\phi)} + \sum_{\mathbf{z}\in Z_{k+1}} \frac{p^\circ_D c_{k+1}(\mathbf{z})}{p^\circ_D N^\circ_{k+1|k} c_{k+1}(\mathbf{z}) + D_{k+1|k}[p_D L_\mathbf{z}]} \tag{36}$$

The corrector equation for the p.g.f. is

$$G_{k+1|k+1}(x) = \frac{x^m \cdot G^{(m)}_{k+1|k}(x\cdot\phi_k)}{G^{(m)}_{k+1|k}(\phi_k)} \tag{37}$$

5 Multisensor CPHD filters (review)

In a companion paper at this conference [1], I demonstrate that there are multisensor CPHD filters that (1) are invariant under sensor reordering, (2) require much weaker simplifying assumptions than the iterated-corrector approach to multisensor CPHD filtering, and (3) are potentially computationally tractable. In this section I summarize the measurement-update equations for this product multisensor CPHD (PM-CPHD) filter. Let notation be as specified in Section 3.1. Then the measurement-updated multisensor PHD and cardinality distribution are, respectively,

$$\overset{1\ldots s}{D}_{k+1|k+1}(\mathbf{x}) = \overset{1\ldots s}{L}_{\overset{1}{Z}_{k+1},\ldots,\overset{s}{Z}_{k+1}}(\mathbf{x}) \cdot \overset{1\ldots s}{s}_{k+1|k}(\mathbf{x}) \tag{38}$$

$$\overset{1\ldots s}{p}_{k+1|k+1}(n) = \frac{\tilde{p}(n)\cdot\sigma^n}{\tilde{G}(\sigma)} \tag{39}$$

where

$$\overset{1\ldots s}{L}_{\overset{1}{Z}_{k+1},\ldots,\overset{s}{Z}_{k+1}}(\mathbf{x}) = \frac{\tilde{G}^{(1)}(\sigma)}{\tilde{G}(\sigma)} \cdot \frac{\overset{1}{L}_{\overset{1}{Z}_{k+1}}(\mathbf{x}) \cdots \overset{s}{L}_{\overset{s}{Z}_{k+1}}(\mathbf{x})}{\overset{1}{N}_{k+1|k+1} \cdots \overset{s}{N}_{k+1|k+1}} \tag{40}$$

$$\sigma = \frac{\overset{1\ldots s}{s}_{k+1|k}[\overset{1}{L}_{\overset{1}{Z}_{k+1}} \cdots \overset{s}{L}_{\overset{s}{Z}_{k+1}}]}{\overset{1}{N}_{k+1|k+1} \cdots \overset{s}{N}_{k+1|k+1}} \tag{41}$$

$$\tilde{G}(x) = \sum\nolimits_{n=0}^{\infty} \tilde{p}(n)\cdot x^n \tag{42}$$

$$\tilde{p}(n) = \overset{1}{\ell}_{\overset{1}{Z}_{k+1}}(n) \cdots \overset{s}{\ell}_{\overset{s}{Z}_{k+1}}(n) \cdot \overset{1\ldots s}{p}_{k+1|k}(n) \tag{43}$$

and where, for $j = 1,\ldots,s$,

$$\overset{j}{\ell}_{\overset{j}{Z}_{k+1}}(n) = \sum_{l=0}^{\min\{n,\overset{j}{m}\}} \overset{j}{C}{}^{(\overset{s}{m}-l)}_{k+1}(0)\cdot l!\cdot C_{n,l} \cdot \overset{1\ldots s}{s}_{k+1|k}[1-\overset{j}{p}_D]^{n-l} \cdot \overset{j}{\sigma}_l(\overset{j}{Z}_{k+1}) \tag{44}$$

$$\overset{j}{L}_{\overset{j}{Z}_{k+1}}(\mathbf{x}) = \overset{j}{\alpha}_0(1-\overset{j}{p}_D(\mathbf{x})) + \sum_{\overset{j}{\mathbf{z}}\in\overset{j}{Z}_{k+1}} \frac{\overset{j}{p}_D(\mathbf{x})\cdot\overset{j}{L}_{\overset{j}{\mathbf{z}}}(\mathbf{x})\cdot\overset{j}{\alpha}(\overset{j}{\mathbf{z}})}{\overset{j}{c}_{k+1}(\overset{j}{\mathbf{z}})} \tag{45}$$

$$\overset{j}{N}_{k+1|k+1} = \overset{j}{\alpha}_0 \, \overset{1\ldots s}{s}_{k+1|k}[1-\overset{j}{p}_D] + \sum_{\overset{j}{\mathbf{z}}\in\overset{j}{Z}_{k+1}} \frac{\overset{1\ldots s}{s}_{k+1|k}[\overset{j}{p}_D \overset{j}{L}_{\overset{j}{\mathbf{z}}}]\cdot\overset{j}{\alpha}(\overset{j}{\mathbf{z}})}{\overset{j}{c}_{k+1}(\overset{j}{\mathbf{z}})} \tag{46}$$

$$\overset{j}{\alpha}_0 = \frac{\sum_{l=0}^{\overset{j}{m}} \overset{j}{C}{}^{(\overset{j}{m}-l)}_{k+1}(0)\cdot\overset{1..s}{G}{}^{(l+1)}_{k+1|k}(\overset{j}{\gamma})\cdot\overset{j}{\sigma}_l(\overset{j}{Z}_{k+1})}{\sum_{i=0}^{\overset{j}{m}} \overset{j}{C}{}^{(\overset{j}{m}-i)}_{k+1}(0)\cdot\overset{1\ldots s}{G}{}^{(i)}_{k+1|k}(\overset{j}{\gamma})\cdot\overset{j}{\sigma}_i(\overset{j}{Z}_{k+1})} \tag{47}$$

$$\overset{j}{\alpha}(\overset{j}{\mathbf{z}}) = \frac{\sum_{l=0}^{\overset{j}{m}-1} \overset{j}{C}{}^{(\overset{j}{m}-l-1)}_{k+1}(0)\cdot\overset{1..s}{G}{}^{(l+1)}_{k+1|k}(\overset{j}{\gamma})\cdot\overset{j}{\sigma}_l(\overset{j}{Z}_{k+1}-\{\overset{j}{\mathbf{z}}\})}{\sum_{i=0}^{\overset{j}{m}} \overset{j}{C}{}^{(\overset{j}{m}-i)}_{k+1}(0)\cdot\overset{1\ldots s}{G}{}^{(i)}_{k+1|k}(\overset{j}{\gamma})\cdot\overset{j}{\sigma}_i(\overset{k}{Z}_{k+1})} \tag{48}$$

$$\overset{j}{\sigma}_i(\overset{j}{Z}_{k+1}) = \sigma_{\overset{j}{m},j}\left(\frac{\overset{1\ldots s}{s}_{k+1|k}[\overset{j}{p}_D \overset{j}{L}_{\overset{j}{\mathbf{z}}_1}]}{\overset{j}{c}_{k+1}(\overset{j}{\mathbf{z}}_1)},\ldots,\frac{\overset{1\ldots s}{s}_{k+1|k}[\overset{j}{p}_D \overset{j}{L}_{\overset{j}{\mathbf{z}}_{\overset{j}{m}}}]}{\overset{j}{c}_{k+1}(\overset{j}{\mathbf{z}}_{\overset{j}{m}})}\right) \tag{49}$$

$$\overset{j}{\gamma} = \overset{1\ldots s}{s}_{k+1|k}[1-\overset{j}{p}_D] \tag{50}$$

6 The "multitarget intensity filter"

At this conference in 2008, R. Streit and L. Stone [12] claimed to have derived a "multitarget intensity filter" using an "elementary" Poisson point process (PPP) approach that includes the PHD filter as a special case and which is capable of estimating the intensity function $\kappa_{k+1}(\mathbf{z})$ of the clutter process. Their claimed derivation employed an "augmented" state space $S^+ = S \cup S_\phi$, where S is the single-target state space and S_ϕ is a space of "clutter targets." Specifically, these authors claimed:

- "[T]he target birth and measurement clutter processes that are assumed specified *a priori* in [Mahler's PHD papers] are estimated here" (p. 1, top of 2nd column).
- "Replacing the estimated clutter intensity with the *a priori* clutter intensity gives the PHD filter..." (p. 6, 1st paragraph).
- The "intensity filter" and thus the PHD filter can be "understood in essentially elementary terms" using "PPP's at an elementary level" (2nd paragraph, Sect. 1).

Explicitly or implicitly, Streit in particular has made the following rather bold claim: Because the PHD filter—and "multitarget intensity filters" in general—require only the elementary "PPP approach," the FISST multitarget calculus is a needless mathematical obfuscation, soon to be swept into the dustbin of history by that approach.

Streit has also claimed [11] that similar "elementary" methods produce a "general multisensor intensity filter," allegedly superior to the iterated-corrector heuristic used for multisensor PHD filtering—thus further demonstrating that the "PPP approach" has supplanted FISST.

Such audacity might be justifiable if the authors' claims were true. They are not. Recently [5] I demonstrated that the "general multisensor intensity filter" is erroneous. It does not reduce to the correct (and intuitively obvious) answers in important special cases, and will perform badly in even the easiest multitarget tracking problems. The reason is that its measurement-update equation involves a sum $\overset{1}{L}_{\overset{1}{Z}_{k+1}}(\mathbf{x})+...+\overset{s}{L}_{\overset{s}{Z}_{k+1}}(\mathbf{x})$ rather than, as in Section 5, a product $\overset{1}{L}_{\overset{1}{Z}_{k+1}}(\mathbf{x})\cdots\overset{s}{L}_{\overset{s}{Z}_{k+1}}(\mathbf{x})$ of pseudo-likelihoods. It is thus incompatible with the assumption of independence of sensors.

In this section I complete this critique. I demonstrate that *even the single-sensor "multitarget intensity filter" is profoundly erroneous*. This demonstration is of clear scientific interest. It exposes the falsity of supposedly rigorous mathematical criticisms of FISST and the PHD filter, and highlights the dangers of eschewing rigorous, FISST-based reasoning. Since it requires unearthing multiple errors buried in a lengthy argument, it is itself necessarily detailed and long. Those readers who do not find it interesting are encouraged to skip to Section 7.

The most serious issues with the "multitarget intensity filter" are as follows:

1. The authors' derivation is not based on some underlying, systematic "PPP approach," as claimed. Rather, it consists of an extemporized sequence of *ad hoc* arguments and hidden assumptions—many of which are seriously flawed and even erroneous—contrived so as to reverse engineer the PHD filter equations.

2. The authors provide no concise description of the "intensity filter." Rather, the reader must ferret out its defining equations, its *a priori* models, and its simplifying assumptions from an involved mathematical argument that stretches over many pages.

3. The only strong simplifying assumption required for the derivation of the PHD filter is that, in the measurement-update step, the predicted multitarget distribution $f_{k+1|k}(X|Z^{(k)})$ must be Poisson. The authors' derivation requires several additional hidden, *ad hoc* assumptions: (a) the distribution $f_{k|k}(X|Z^{(k)})$ is Poisson;[1] (b) targets do not spawn other targets;[2] (c) the intensity function of the target-birth process is constant;[3] (d) the posterior detected-target distribution is approximately Poisson;[4] (e) targets are well-separated relative to sensor resolution;[5] and (f) S is bounded.[6] Thus even if their derivation were not erroneous, the authors would not have actually succeeded in re-deriving the PHD filter in an "elementary" fashion, as claimed.

4. In particular, the hidden assumption that targets are well-separated negates the authors' claim to have a "multitarget" filter.

5. The only way that new actual targets can appear is when clutter targets transition to actual targets.[7] This model of target birth is phenomenologically erroneous.

6. Certain steps in the derivation of the "intensity filter" have no evident justification and some are erroneous.

7. In particular, the authors' derivation of their corrector equation involves a fatal mathematical error, with the consequence that the "intensity filter" is not valid for all measurement collections.

8. The "intensity filter" cannot, as claimed, estimate the intensity function of the clutter process.

9. Nor can it, as claimed, estimate the intensity function of the target-appearance process—or even the birth rate.

10. If one applies the PHD and CPHD filter equations to the space $S = S \cup S_\phi$ and applies some simple algebra, one gets *non-erroneous, theoretically rigorous* "multitarget intensity filters." Indeed, the most general "clutter-agnostic" PHD filter described in [9] *does* estimate the clutter intensity function. The simplest filter described in [2,9], and in Section 4.2, is what results if one replaces the complex and erroneous "PPP approach" in [12] with a simple, rigorous argument. This means that the authors' "PPP approach" is not only misconceived, but a needless mathematical obfuscation.

In the remainder of this section I provide greater detail to substantiate these claims.

Item 5: The authors use a Markov transition—from clutter targets to actual targets—to model the birth of actual targets. This model is phenomenologically erroneous. Suppose that the clutter statistics are very unlike the target statistics. Exploiting this difference would make it easier to distinguish targets from clutter. But if clutter targets can transition to actual targets, then clutter statistics are inherently intermixed with target statistics—thus making it more difficult, not less difficult, to distinguish targets from clutter.

Though the authors do not do so in [12], note that it would be equally erroneous to model target deaths as a Markov transition from actual targets to clutter targets.

Correct target birth and death modeling follows from an elementary fact about point processes. A point process on the single-target space S is a random finite subset Ξ of S. The null instantiation $\Xi = \emptyset$ of Ξ represents the possibility that no targets are present. That is, it represents *target non-existence*. A Markov transition from one multitarget state X' to another multitarget state X includes the possibility of transition from X' to $\emptyset$—i.e, of *target disappearance*. It also includes the possibility of transition from $\emptyset$ to X—i.e., of *target appearance*. In particular: the appearance of a single target is a transition from the null state $X' = \emptyset$ to a singleton state $X = \{\mathbf{x}\}$—not from a clutter generator ϕ to a target $\mathbf{x}$.

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[1] Their Eq. (5) in [11].

[2] Implicit in their Eq. (8).

[3] Implicit in their Eqs. (40,41).

[4] Their Eq. (22).

[5] Implicit in their Eq. (26).

[6] Implicit in their Eq. (37) if $c_4 \neq 0$. Also since the target-birth spatial distribution must be uniform because of Item 3-(c).

[7] Implicit in their Eq. (37) since $c_1 \neq 0$, where c_1 is the probability (density) that a clutter target will transition to an actual target.

Items 1-4, 6, 7: The authors' derivation of their corrector equation involves a fatal mathematical error.

To show this, I enter their derivation at their Eq. (20) in [12]. Let $Z_k = \{\mathbf{z}_1, \ldots, \mathbf{z}_m\}$ with $|Z_k| = m$ be the measurement-set collected at time-step k, and let $\xi = (n, \mathbf{x}_1, \ldots, \mathbf{x}_n)$ be the multitarget state (in a vector notation). Then the *posterior detected-targets distribution* is

$$f^d(\xi) = \frac{\delta_{m,n}}{n!} \sum_{\sigma} \prod_{j=1}^{n} \frac{f_k(\mathbf{z}_{\sigma j} \mid \mathbf{x}_j) \cdot p_{D,k}(\mathbf{x}_j) \cdot D^d_{k|k-1}(\mathbf{x}_j)}{\lambda^T_{k|k-1}(\mathbf{z}_{\sigma j})} \qquad (20^*)$$

where $D^d_{k|k-1}(\mathbf{x})$ is the predicted detected-target PHD and

$$\lambda^T_{k|k-1}(\mathbf{z}) = \int f_k(\mathbf{z} \mid \mathbf{x}) \cdot p_{D,k}(\mathbf{x}) \cdot D^d_{k|k-1}(\mathbf{x}) d\mathbf{x} \qquad (15^*)$$

(The authors only implicitly include the factor $\delta_{m,n}$. Also, I have written (20*) and (15*) rather than (20) and (15) to avoid confusion.) In the first paragraph of their Section 6.2, the authors note that Eq. (20*) cannot be the multitarget distribution of a Poisson process. So they propose to replace it with its Poisson approximation,[8]

$$f^d(\xi) \cong \frac{e^{-c}}{n!} \prod_{s=1}^{n} D^d_{k|k}(\mathbf{x}_s) \qquad (51)$$

where $D^d_{k|k}(\mathbf{x})$ is the intensity function (PHD) of $f^d(\xi)$ and c is the integral of $D^d_{k|k}(\mathbf{x})$.

The authors must determine $D^d_{k|k}(\mathbf{x})$. This follows easily from multitarget calculus: $D^d(\mathbf{x}) = \int f^d(\{\mathbf{x}\} \cup Y) \delta Y = m \cdot f^d(\mathbf{x}|m)$ where $f^d(\mathbf{x}|m)$ is the marginal for $n = m$,

$$f^d(\mathbf{x} \mid m) = \int f^d(m, \mathbf{x}, \mathbf{x}_1, \ldots, \mathbf{x}_{m-1}) d\mathbf{x}_1 \cdots d\mathbf{x}_{m-1} \qquad (21^*)$$

But the authors claim that multitarget calculus is superfluous. So instead they assume that, when $n = m$,

$$f^d(n, \mathbf{x}_1, \ldots, \mathbf{x}_n) \cong \prod_{s=1}^{n} f^d(\mathbf{x}_s \mid m) \qquad (26^*)$$

This unrealistic approximation negates their claim to have a "multitarget" filter.[9] The marginal turns out to be:

$$f^d(\mathbf{x} \mid m) = \frac{1}{m} \sum_{r=1}^{m} \frac{f_{k+1}(\mathbf{z}_r \mid \mathbf{x}) \cdot p_{D,k}(\mathbf{x}) \cdot D^d_{k|k-1}(\mathbf{x})}{\lambda^T_{k|k-1}(\mathbf{z}_r)} \qquad (22^*)$$

The authors then claim, without substantiation, that[10]

$$D^d_{k|k}(\mathbf{x}) = c \cdot f^d(\mathbf{x} \mid m) \qquad (27^*)$$

from which follows

$$D^d_{k|k}(\mathbf{x}) = \frac{c}{m} \sum_{r=1}^{m} \frac{f_{k+1}(\mathbf{z}_r \mid \mathbf{x}) \cdot p_{D,k}(\mathbf{x}) \cdot D^d_{k|k-1}(\mathbf{x})}{\lambda^T_{k|k-1}(\mathbf{z}_r)} \qquad (52)$$

If they can show that $c = m$ then they will have managed to reverse-engineer the correct formula for $D^d_{k|k}(\mathbf{x})$. To this end they claim, without substantiation, that the right-hand side of Eq. (51) is the "likelihood" for c, conditioned on ξ with $n = m$:[11]

$$L(c \mid \xi) = \frac{e^{-c}}{m!} \prod_{s=1}^{m} D^d_{k|k}(\mathbf{x}_s) \propto e^{-c} \cdot c^m \qquad (28^*)$$

Finally they claim, without substantiation, that the value of c can be obtained as the maximum likelihood estimate:

$$\hat{c}_{ML} = \arg\sup_c L(c \mid X) = m. \qquad (53)$$

These last two claims result in a fatal error. The "intensity filter," no less than any multitarget filter, must be valid for each and every instantiation of the current measurement-set Z_k. But this cannot be the case. First note that the only way that c can have a likelihood function is if it is a random number. Because of Eq. (27*),

$$c = c(Z_k) \qquad (54)$$

is a function of the random variable Z_k. It is not possible for $c(Z_k)$ to be a constant function of Z_k. If it were, it would have a likelihood function of the form

$$L(c|X) = \delta_{c'}(c) \qquad (55)$$

for some c', where $\delta_{c'}(c)$ is the Dirac delta function concentrated at c'. But according to Eq. (28*) its likelihood does not have this form. Thus $c(Z_k)$ must be a non-constant function of Z_k. Thus there will be instantiations of Z_k such that $c(Z_k) \neq m$. (If otherwise, $c(Z_k)$ would be constant.) Thus the "intensity filter" cannot be valid for all possible measurement-collections. The reason: Eqs. (28*,53) are erroneous improvisations.

Item 8: The authors claim that the following is the "estimated clutter intensity [function]":

$$\lambda^C_{k|h}(\mathbf{z}) = f_{k+1}(\mathbf{z} \mid \phi) \cdot N^{Ext}_{k|k-1}(\phi) \qquad (44^*)$$

Here,

$$f_{k+1}(\mathbf{z} \mid \phi) = c_{k+1}(\mathbf{z}) \qquad (56)$$

is the likelihood that a clutter generator ϕ will generate measurement $\mathbf{z}$—i.e., it is the clutter spatial density. Also, $N^{Ext}_{k|k-1}(\phi)$ is the "predicted number of targets in the clutter space". Either $f_{k+1}(\mathbf{z} \mid \phi)$ is known *a priori* or it is not. If the former then the spatial distribution of $\lambda^C_{k|h}(\mathbf{z})$ is known—contrary to the claim that the "intensity filter" can estimate the *entire* clutter intensity. If $f_{k+1}(\mathbf{z} \mid \phi)$ is not known, then $\lambda^C_{k|h}(\mathbf{z})$ is also not known, again contrary to claim. Eq. (44*) shows only that, were the "intensity filter" non-erroneous, at best it would be able to estimate only the clutter rate, $N^{Ext}_{k|k-1}(\phi)$.

Item 9: The authors initially claim to be able to estimate the target birth process, but end up providing a formula for only the "estimated predicted birth rate":

$$\hat{b}_k(\mathbf{x}) = \int_{S_\phi} f^{Ext}_{k|k-1}(\mathbf{x} \mid \mathbf{x}') \cdot p^{Ext}_{S,k|k-1}(\mathbf{x}') \cdot D^{Ext}_{k-1|k-1}(\mathbf{x}') d\mathbf{x}' \qquad (41^*)$$

Here, $\mathbf{x}' \in S_\phi$, $f^{Ext}_{k|k-1}(\mathbf{x} \mid \mathbf{x}')$ is the Markov density on $S^+ = S \cup S_\phi$, $p^{Ext}_{S,k|k-1}(\mathbf{x}') = p^{\circ}_{S,k+1|k}$ is the constant probability of target survival on S^+, and $D^{Ext}_{k-1|k-1}(\mathbf{x})$ is the PHD on S^+, known inductively to be constant. For $\mathbf{x} \in S$ and $\mathbf{x}' \in S_\phi$, the authors define $f^{Ext}_{k|k-1}(\mathbf{x} \mid \mathbf{x}') = c_1$, a constant that

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[8] This is an extemporized assumption that is not made in the derivation of the PHD filter. It is a poor approximation since $f^d(X) = 0$ if $n \neq m$.

[9] This is an extemporized assumption that is not made in the derivation of the PHD filter. It is highly unrealistic since it means that the targets must be well-separated with respect to sensor resolution.

[10] The authors do not justify this claim, which would appear to require the FISST multitarget calculus to substantiate.

[11] The claim that $L(c|X)$ is a likelihood function—and that it is the likelihood for c—has no evident justification. For one thing, it cannot be a likelihood function since it does not integrate to 1.

defines the probability (density) that a clutter target will transition to an actual target. Thus Eq. (41*) becomes

$$\hat{b}_k(\mathbf{x}) = c_1 \cdot p^{\circ}_{S,k|k-1} \cdot \int_{S_\phi} D^{Ext}_{k-1|k-1}(\mathbf{x}')d\mathbf{x}' \tag{41*}$$

$$= c_1 \cdot p^{\circ}_{S,k|k-1} \cdot N^{Ext}_{k-1|k-1}(\phi) \tag{57}$$

Thus the birth process has a constant intensity function—a severe restriction, and one that is possible only if S is bounded. Also, Eq. (41*) cannot be an estimate of the target-birth rate. As noted in Item 5, if $c_1 \neq 0$ then target and clutter statistics are intermingled—and thus Eq. (41*) cannot be the birth rate for actual targets exclusively.

7 Mathematical derivations

The single-sensor LC-CPHD filter is a special case of the clutter-agnostic CPHD filter of Section 4.2. It is based on the following assumptions:

- $p^{\circ}_{S,k} = 0$—i.e., no clutter generator survives from time-step k into time-step k+1.
- $p^{\circ}_D = 1$—i.e., every clutter generator generates a measurement at time-step k+1.
- Clutter generators appear spontaneously at time-step k+1. The spatial distribution of the measurements they generate, $c_{k+1}(\mathbf{z})$, is known *a priori*. Their p.g.f. is $B^{\circ}_{k+1|k}(x) = C_{k+1}(x)$ where $C_{k+1}(z)$ is the known p.g.f. of the clutter-measurement process and $B^{\circ}_{k+1|k}(x)$ is the p.g.f. of the birth process for the clutter generators.

In particular, from Eq. (31),

$$N^{\circ}_{k+1|k} = N^{\circ}_{B,k+1|k} + p^{\circ}_{S,k+1|k} \cdot N^{\circ}_{k|k} \tag{58}$$

$$= N^{\circ}_{B,k+1|k} = \lambda_{k+1} \tag{59}$$

Thus from Eq. (34),

$$\frac{N^{\circ}_{k+1|k+1}}{N^{\circ}_{k+1|k}} = \frac{1-p^{\circ}_D}{N_{k+1|k} + N^{\circ}_{k+1|k}} \cdot \frac{G^{(m+1)}_{k+1|k}(\phi_k)}{G^{(m)}_{k+1|k}(\phi_k)} + \sum_{\mathbf{z}\in Z_{k+1}} \frac{p^{\circ}_D c_{k+1}(\mathbf{z})}{p^{\circ}_D N^{\circ}_{k+1|k} c_{k+1}(\mathbf{z}) + D_{k+1|k}[p_D L_{\mathbf{z}}]} \tag{60}$$

and so

$$N^{\circ}_{k+1|k+1} = \sum_{\mathbf{z}\in Z_{k+1}} \frac{\lambda_{k+1} c_{k+1}(\mathbf{z})}{\lambda_{k+1} c_{k+1}(\mathbf{z}) + D_{k+1|k}[p_D L_{\mathbf{z}}]} \tag{61}$$

This equation has no effect on subsequent filter steps since

$$N^{\circ}_{k+2|k+1} = N^{\circ}_{B,k+2|k+1} + p^{\circ}_{S,k+2|k+1} \cdot N^{\circ}_{k+1|k+1} = \lambda_{\kappa+2} \tag{62}$$

In other words, the filter for $N^{\circ}_{k|k}$ has been disabled. The only two filtering equations that remain are those for the PHD and the p.g.f.

Given this, I derive the predictor equations and corrector equations for the LC-CPHD filter in, respectively, Section 7.1 and 7.2.

7.1 Predictor equations (single-sensor)

From Eq. (30) the predictor equation for the PHD is:

$$D_{k+1|k}(\mathbf{x}) = b_{k+1|k}(\mathbf{x}) + \int p_{S,k}(\mathbf{x}') \cdot f_{k+1|k}(\mathbf{x}\,|\,\mathbf{x}') \cdot D_{k|k}(\mathbf{x})d\mathbf{x} \tag{63}$$

From Eqs. (32,33) the predictor equation for the p.g.f. is

$$G_{k+1|k}(x) = B_{k+1|k}(x) \cdot B^{\circ}_{k+1|k}(x) \cdot G_{k|k}(1-\psi_k + \psi_k \cdot x) \tag{64}$$

$$= B_{k+1|k}(x) \cdot C_{k+1}(x) \cdot G_{k|k}(1-\psi_k + \psi_k \cdot x) \tag{65}$$

where

$$\psi_k = \frac{D_{k|k}[p_{S,k}] + p^{\circ}_{S,k} N^{\circ}_{k|k}}{N_{k|k} + N^{\circ}_{k|k}} \tag{66}$$

$$= \frac{D_{k|k}[p_{S,k}]}{G^{(1)}_{k|k}(1)} \tag{67}$$

7.2 Corrector equations (single-sensor)

First note that, from Eq. (31),

$$N^{\circ}_{k+1|k} = N^{\circ}_{B,k+1|k} + p^{\circ}_{S,k+1|k} \cdot N^{\circ}_{k|k} = N^{\circ}_{B,k+1|k} = \lambda_{k+1} \tag{68}$$

So, from Eqs. (34,35) the PHD corrector equation is

$$\frac{D_{k+1|k+1}(\mathbf{x})}{D_{k+1|k}(\mathbf{x})} = \frac{1-p_D(\mathbf{x})}{N_{k+1|k} + N^{\circ}_{k+1|k}} \cdot \frac{G^{(m+1)}_{k+1|k}(\phi_k)}{G^{(m)}_{k+1|k}(\phi_k)} + \sum_{\mathbf{z}\in Z_{k+1}} \frac{p_D(\mathbf{x}) \cdot L_{\mathbf{z}}(\mathbf{x})}{p^{\circ}_D N^{\circ}_{k+1|k} c_{k+1}(\mathbf{z}) + D_{k+1|k}[p_D L_{\mathbf{z}}]} \tag{69}$$

$$= \frac{1-p_D(\mathbf{x})}{N_{k+1|k} + \lambda_{k+1}} \cdot \frac{G^{(m+1)}_{k+1|k}(\phi_k)}{G^{(m)}_{k+1|k}(\phi_k)} + \sum_{\mathbf{z}\in Z_{k+1}} \frac{p_D(\mathbf{x}) \cdot L_{\mathbf{z}}(\mathbf{x})}{\lambda_{k+1} c_{k+1}(\mathbf{z}) + D_{k+1|k}[p_D L_{\mathbf{z}}]} \tag{70}$$

where

$$\phi_k = \frac{D_{k+1|k}[1-p_D] + N^{\circ}_{k+1|k}(1-p^{\circ}_D)}{N_{k+1|k} + N^{\circ}_{k+1|k}} \tag{71}$$

$$= \frac{D_{k+1|k}[1-p_D]}{N_{k+1|k} + \lambda_{k+1}} \tag{72}$$

From Eq. (37) the corrector equation for the p.g.f. is

$$G_{k+1|k+1}(x) = \frac{x^m \cdot G^{(m)}_{k+1|k}(x \cdot \phi_k)}{G^{(m)}_{k+1|k}(\phi_k)} \tag{73}$$

We also need the corrector equation for the cardinality distribution. This is given by

$$p_{k+1|k+1}(n) = \frac{1}{n!} \frac{d^n G_{k+1|k+1}}{dx^n}(0) \tag{74}$$

However,

$$G^{(m)}_{k+1|k}(x) \cdot \frac{d^n G_{k+1|k+1}}{dx^n}(x) = \frac{d^n}{dx^n}\left(x^m \cdot G^{(m)}_{k+1|k}(x \cdot \phi_k)\right) \tag{75}$$

$$= \sum_{i=0}^{n} C_{n,i} \cdot \left(\frac{d^i}{dx^i} x^m\right)\left(\frac{d^{n-i}}{dx^{n-i}} G^{(m)}_{k+1|k}(x \cdot \phi_k)\right) \tag{76}$$

$$= \sum_{i=0}^{n} C_{n,i} \cdot i! \cdot C_{m,i} \cdot x^{m-i} \cdot G^{(m+n-i)}_{k+1|k}(x \cdot \phi_k) \cdot \phi_k^{n-i} \tag{77}$$

Thus setting $x = 0$,

$$p_{k+1|k+1}(n) = \frac{1}{n!} \cdot \frac{C_{n,m} \cdot G^{(n)}_{k+1|k}(0) \cdot \phi_k^{n-m}}{G^{(m)}_{k+1|k}(\phi_k)} \tag{78}$$

$$\propto \ell_{Z_{k+1}}(n) \cdot p_{k+1|k}(n) \tag{79}$$

where

$$\ell_{Z_{k+1}}(n) = C_{n,m} \cdot \phi_k^{n-m} \tag{80}$$

7.3 Predictor equations (multisensor)

Suppose that we have s sensors, each with its own space of clutter generators. Then the multisensor analogs of the predictor equations

$$D_{k+1|k}(\mathbf{x}) = b_{k+1|k}(\mathbf{x}) + \int p_{S,k}(\mathbf{x}') \cdot f_{k+1|k}(\mathbf{x} \mid \mathbf{x}') \cdot D_{k|k}(\mathbf{x}) d\mathbf{x} \tag{81}$$

$$N^{\circ}_{k+1|k} = N^{\circ}_{B,k+1|k} + p^{\circ}_{S,k+1|k} \cdot N^{\circ}_{k|k} \tag{82}$$

in Section 4.2 are the predictor equations

$$D_{k+1|k}(\mathbf{x}) = b_{k+1|k}(\mathbf{x}) + \int p_{S,k}(\mathbf{x}') \cdot f_{k+1|k}(\mathbf{x} \mid \mathbf{x}') \cdot D_{k|k}(\mathbf{x}) d\mathbf{x} \tag{83}$$

$$\overset{j}{N}{}^{\circ}_{k+1|k} = \overset{j}{N}{}^{\circ}_{B,k+1|k} + \overset{j}{p}{}^{\circ}_{S,k+1|k} \cdot \overset{j}{N}{}^{\circ}_{k|k} \qquad (j = 1,\ldots,s) \tag{84}$$

$$G_{k+1|k}(x) = B_{k+1|k}(x) \cdot \overset{1}{B}{}^{\circ}_{k+1|k}(x) \cdots \overset{s}{B}{}^{\circ}_{k+1|k}(x) \cdot G_{k|k}(1 - \psi_k + \psi_k \cdot x) \tag{85}$$

$$\psi_k = \frac{D_{k|k}[p_{S,k}] + \sum_{j=1}^{s} \overset{j}{p}{}^{\circ}_{S,k+1|k} \overset{j}{N}{}^{\circ}_{k|k}}{N_{k|k} + \sum_{j=1}^{s} \overset{j}{N}{}^{\circ}_{k|k}} \tag{86}$$

where the notation is self-explanatory.

Assume that $\overset{j}{p}{}^{\circ}_{S,k+1|k} = 0$, $\overset{j}{p}{}^{\circ}_{D,k+1} = 1$ $\overset{j}{N}{}^{\circ}_{B,k+1|k} = \overset{j}{\lambda}_{k+1}$, $\overset{j}{B}{}^{\circ}_{k+1|k}(x) = \overset{j}{C}_{k+1}(x)$, $\overset{j}{C}{}^{(1)}_{k+1}(1) = \overset{j}{\lambda}_{k+1}$ for all $j = 1,\ldots,s$. Then we get, as claimed,

$$D_{k+1|k}(\mathbf{x}) = b_{k+1|k}(\mathbf{x}) + \int p_{S,k}(\mathbf{x}') \cdot f_{k+1|k}(\mathbf{x} \mid \mathbf{x}') \cdot D_{k|k}(\mathbf{x}) d\mathbf{x} \tag{87}$$

$$G_{k+1|k}(x) = B_{k+1|k}(x) \cdot \overset{1}{C}_{k+1}(x) \cdots \overset{s}{C}_{k+1}(x) \cdot G_{k|k}(1 - \psi_k + \psi_k \cdot x) \tag{88}$$

$$\psi_k = \frac{D_{k|k}[p_{S,k}]}{N_{k|k} + \sum_{j=1}^{s} \overset{j}{N}{}^{\circ}_{k|k}} = \frac{D_{k|k}[p_S]}{G^{(1)}_{k|k}(1)} \tag{89}$$

7.4 Corrector equations (multisensor)

Eqs. (17-26) follow from Eqs. (38-50). Eqs. (47-50) are no longer needed since Eqs. (44,46) are replaced by the multisensor versions of Eqs. (4,8). Eq. (45) results from multiplying Eq. (4) by $D_{k+1|k}(\mathbf{x})$ and integrating over $\mathbf{x}$.

8 Conclusion

I have derived a CPHD filter with computational complexity $O(mn)$, given that the measurement process of a clutter generator is Bernoulli. Future research must determine if this restriction leads to reduced performance.

The new linear-complexity CPHD (LC-CPHD) filter is a special case of the "clutter-agnostic" CPHD filters described in [2,9].

I also derived formulas for a multisensor version of the LC-CPHD filter. It appears that this multisensor filter has computational complexity $O(\overset{1}{m} \cdots \overset{s}{m} n)$, where $\overset{j}{m}$ is the number of measurements collected by the j^{th} sensor.

Finally, I demonstrated that the "multitarget intensity filter" [12] is profoundly erroneous. Moreover, the simplest filter described in [2,9] (or Section 4.2) is what results when one replaces the erroneous "PPP approach" in [12] with a rigorous, genuinely elementary derivation.

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